

FREIDLIN-WENTZELL'S LARGE DEVIATIONS FOR STOCHASTIC EVOLUTION EQUATIONS

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ABSTRACT. We prove a Freidlin-Wentzell large deviation principle for general stochastic evolution equations with small perturbation multiplicative noises. In particular, our general result can be used to deal with a large class of quasi linear stochastic partial differential equations, such as stochastic porous medium equations and stochastic reaction diffusion equations with polynomial growth zero order term and p -Laplacian second order term.

1. INTRODUCTION

Since the work of Freidlin and Wentzell [14], the theory of small perturbation large deviations for stochastic differential equations(SDE) has been extensively developed(cf. [2, 30], etc.). In classical method, to establish such a large deviation principle(LDP) for SDEs, one needs to discretize the time variable and then prove various necessary exponential continuity and tightness for stochastic dynamical systems in different spaces by using comparison principle. However, such verifications would become rather complicated and even impossible in some cases for infinite stochastic partial differential equations with multiplicative noises.

Recently, Dupuis and Ellis [11] systematically developed a weak convergence approach to the theory of large deviation. The core idea is to prove some variational representation formula about the Laplace transform of bounded continuous functionals, which will lead to proving an equivalent Laplace principle with LDP. In particular, for Brownian functionals, an elegant variational representation formula has been established by Boué-Dupuis [3] and Budhiraja-Dupuis [5]. A simplified proof is given by the second named author [32]. This variational representation has been proved to be very effective for various finite dimensional stochastic dynamical system with irregular coefficients(cf. [4, 23, 24], etc.). One of the main advantages of this argument is that one only needs to make some necessary moment estimates. This can be seen completely from the present paper that it also works very well for infinite dimensional stochastic dynamical systems.

In the past two decades, there are numerous results about the LDP for stochastic partial differential equations(SPDE) (cf. [29, 10, 20, 16, 7, 12, 6], etc.). All these results are concentrated on semi-linear SPDEs, i.e., the second order term is linear, and their proofs, except [12, 6], are mainly based on the classical exponential tightness method. In [12], the approach for LDP is based on nonlinear semigroup and infinite dimensional

Key words and phrases. Laplace Principle, Freidlin-Wentzell's Large Deviation, Stochastic Evolution Equation, Variational Representation Formula.

Hamilton-Jacobi equations. The approach in [6] is based on the variational representation. Recently, Röckner-Wang-Wu [26] proved an LDP for stochastic porous medium equation with additive noise by using the classical comparison principle. It should be pointed out that the equation of this type has a non-linear and degenerated second order term. Since additive noise was considered in [26], they can discretize time and prove some necessary estimates. It seems difficult to extend their result to the multiplicative noise case by using the classical method.

On the other hand, the existence and uniqueness of SPDEs have already been studied in various literatures prior to LDP for SPDEs(cf. [8, 18, 27, 10, 15, 31], etc.). In the theory of SPDEs, there exist two main tools: semigroup method and variation method(or monotone method). One of the merits of semigroup method is that the noise can take values in a larger space(cf. [10]). But, it can only deal with semi-linear SPDEs. The variation method combined with Galerkin's approximation is usually used in the framework of evolution triple(cf. [18, 31]). Thus, as in the deterministic case(cf. [28]), it can tackle a large class of SPDEs. But, the diffusion coefficients need to be in the space of Hilbert-Schmidt operators.

Our aim in the present paper is to prove a Freidlin-Wentzell's large deviation for stochastic evolution equations in the evolution triple case by using the weak convergence approach as done in [6]. Thus, the main point is to prove the tightness of some control stochastic evolution equations. This will be realized by making some moment estimates in suitable space(see Lemma 3.2 below) and then using the general tightness criterion for stochastic processes(see Lemma 3.4 below). Moreover, in order to treat the SPDEs with polynomial growth, we will work in the framework of [31], which is a little different from [18]. Compared with the well-known results, our proof is succinct, and we believe that our method can be adapted to some other non-linear stochastic equations such as stochastic Navier-Stokes equation.

This paper is organized as follows: In Section 2, we shall give our framework and recall an abstract criterion for Laplace principle due to Budhiraja-Dupuis [5], as well as an existence and uniqueness result for stochastic evolution equation essentially due to Krylov-Rozovskii [18]. In Section 3, we first prove a Laplace principle for stochastic evolution equation(see Theorem 3.5 below) without any compact embedding requirement. In order to prove the corresponding rate function is good, we need an extra compact assumption (see Lemma 3.7 below). Lastly, in Section 4 we give three applications.

2. FRAMEWORK AND PRELIMINARIES

Let \mathbb{X} be a reflexive and separable Banach space, which is densely and continuously injected in a separable Hilbert space \mathbb{H} . Identifying \mathbb{H} with its dual we get

$$\mathbb{X} \subset \mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{X}^*,$$

where the star “ $*$ ” denotes the dual spaces.

Assume that the norm in \mathbb{X} is given by

$$\|x\|_{\mathbb{X}} := \|x\|_{1,\mathbb{X}} + \|x\|_{2,\mathbb{X}}, \quad x \in \mathbb{X}.$$

Denote by \mathbb{X}_i , $i = 1, 2$ the completions of \mathbb{X} with respect to the norms $\|\cdot\|_{i,\mathbb{X}} =: \|\cdot\|_{\mathbb{X}_i}$. Then $\mathbb{X} = \mathbb{X}_1 \cap \mathbb{X}_2$. Let us also assume that both spaces are reflexive and embedded in \mathbb{H} . Thus, we get two triples:

$$\mathbb{X}_1 \subset \mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{X}_1^*, \quad \mathbb{X}_2 \subset \mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{X}_2^*.$$

Noticing that \mathbb{X}_1^* and \mathbb{X}_2^* can be thought as subspaces of \mathbb{X}^* , one may define a Banach space $\mathbb{Y} := \mathbb{X}_1^* + \mathbb{X}_2^* \subset \mathbb{X}^*$ as follows: $f \in \mathbb{Y}$ if and only if $f = f_1 + f_2$, $f_i \in \mathbb{X}_i^*$, $i = 1, 2$

and the norm of f is defined by

$$\|f\|_{\mathbb{Y}} = \inf_{f=f_1+f_2} (\|f_1\|_{\mathbb{X}_1^*} + \|f_2\|_{\mathbb{X}_2^*}).$$

In the following, the dual pairs of $(\mathbb{X}, \mathbb{X}^*)$ and $(\mathbb{X}_i, \mathbb{X}_i^*)$, $i = 1, 2$ are denoted respectively by

$$[\cdot, \cdot]_{\mathbb{X}}, \quad [\cdot, \cdot]_{\mathbb{X}_i}, \quad i = 1, 2.$$

Then, for any $x \in \mathbb{X}$ and $f = f_1 + f_2 \in \mathbb{Y} \subset \mathbb{X}^*$,

$$[x, f]_{\mathbb{X}} = [x, f_1]_{\mathbb{X}_1} + [x, f_2]_{\mathbb{X}_2}.$$

We remark that if $f \in \mathbb{H}$ and $x \in \mathbb{X}$, then

$$[x, f]_{\mathbb{X}} = [x, f]_{\mathbb{X}_1} = [x, f]_{\mathbb{X}_2} = \langle x, f \rangle_{\mathbb{H}}, \quad (1)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ stands for the inner product in \mathbb{H} .

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a complete separable filtration probability space, and Q a nonnegative definite and symmetric trace operator defined on another separable Hilbert space \mathbb{U} . A Q -Wiener process $\{W(t), t \geq 0\}$ defined on (Ω, \mathcal{F}, P) is given and assumed to be adapted to $(\mathcal{F}_t)_{t \geq 0}$ (cf. [10]). Set $\mathbb{U}_Q := Q^{1/2}(\mathbb{U})$ and let $L_2(\mathbb{U}_Q, \mathbb{H})$ denote the Hilbert space consisting of all Hilbert-Schmidt operators from \mathbb{U}_Q to \mathbb{H} , where the inner product is denoted by $\langle \cdot, \cdot \rangle_{L_2(\mathbb{U}_Q, \mathbb{H})}$, and the norm by $\|\cdot\|_{L_2(\mathbb{U}_Q, \mathbb{H})}$.

In the following, we will work in the finite time interval $[0, T]$. For a Banach space \mathbb{B} we shall denote by $\mathbb{C}_T(\mathbb{B})$ the continuous functions space from $[0, T]$ to \mathbb{B} , which is endowed with the uniform norm. Define

$$\mathbb{L}_Q := \left\{ h = \int_0^\cdot \dot{h}(s) ds : \dot{h} \in L^2(0, T; \mathbb{U}_Q) \right\}$$

with the norm

$$\|h\|_{\mathbb{L}_Q} := \left(\int_0^1 \|\dot{h}(s)\|_{\mathbb{L}_Q}^2 ds \right)^{1/2},$$

where the dot denotes the generalized derivative. Let μ_Q be the law of the Q -Wiener process W in $\mathbb{C}_T(\mathbb{U})$. Then

$$(\mathbb{C}_T(\mathbb{U}), \mathbb{L}_Q, \mu_Q)$$

forms an abstract Wiener space.

For $N > 0$ we set $D_N := \{h \in \mathbb{L}_Q : \|h\|_{\mathbb{L}_Q} \leq N\}$. Then D_N is metrizable as a compact Polish space with respect to the weak topology in \mathbb{L}_Q . Let \mathcal{A}_N denote all continuous and \mathcal{F}_t -adapted process h from $[0, T]$ to \mathbb{U}_Q such that for almost all ω , $h(\cdot, \omega) \in D_N$, i.e.,

$$\int_0^T \|\dot{h}(s, \omega)\|_{\mathbb{U}_Q}^2 ds \leq N. \quad (2)$$

Let \mathbb{S} be a Polish space. A function $I : \mathbb{S} \rightarrow [0, \infty]$ is given.

Definition 2.1. *The function I is called a rate function if I is lower semicontinuous. The function I is called a good rate function if for every $a < \infty$, $\{f \in \mathbb{S} : I(f) \leq a\}$ is compact.*

Let $Z^\varepsilon : \mathbb{C}_T(\mathbb{U}) \rightarrow \mathbb{S}$ be a family of measurable mappings. We assume that

(Hypothesis): There is a measurable map $Z^0 : \mathbb{L}_Q \mapsto \mathbb{S}$ such that for any $N > 0$, if a family $\{h^\varepsilon\} \subset \mathcal{A}_N$ (as random variables in D_N) converges in distribution to a $v \in \mathcal{A}_N$, then for some subsequence ε_k , $Z^{\varepsilon_k}(\cdot + \frac{h^{\varepsilon_k}(\cdot)}{\sqrt{\varepsilon_k}})$ converges in distribution to $Z^0(v)$ in \mathbb{S} .

For each $f \in \mathbb{S}$, define

$$I(f) := \frac{1}{2} \inf_{\{h \in \mathbb{L}_Q: f=Z^0(h)\}} \|h\|_{\mathbb{L}_Q}^2, \quad (3)$$

where $\inf \emptyset = \infty$ by convention.

We recall the following result due to [3, 5] (see also [32, Theorem 4.4]).

Theorem 2.2. $\{Z^\varepsilon, \varepsilon \in (0, 1)\}$ satisfies the Laplace principle with the rate function $I(f)$ given by (3). That is, for each real bounded continuous function g on \mathbb{S} :

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left(\exp \left[-\frac{g(Z^\varepsilon)}{\varepsilon} \right] \right) = - \inf_{f \in \mathbb{S}} \{g(f) + I(f)\}. \quad (4)$$

Remark 2.3. If I in (4) is not lower semicontinuous, then the regularization of I

$$\tilde{I}(f) := \lim_{\varepsilon \downarrow 0} \inf_{f' \in B_\varepsilon(f)} I(f')$$

still satisfies (4), where $B_\varepsilon(f)$ is the ball in \mathbb{S} with center f and radius ε . Moreover, if I is a good rate function, then the Laplace principle is equivalent to the large deviation principle (cf. [11, Theorem 1.2.3]).

We now introduce three evolution operators used in the present paper (cf. [31]):

$$A_i : [0, T] \times \mathbb{X}_i \rightarrow \mathbb{X}_i^* \in \mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{X}_i)/\mathcal{B}(\mathbb{X}_i^*), \quad i = 1, 2,$$

and

$$B : [0, T] \times \mathbb{H} \rightarrow L_2(\mathbb{U}_Q, \mathbb{H}) \in \mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{H})/\mathcal{B}(L_2(\mathbb{U}_Q, \mathbb{H})).$$

In the following, for the sake of simplicity, we write

$$A = A_1 + A_2 \in \mathbb{Y} \subset \mathbb{X}^*,$$

and assume throughout this paper that

(H1) (Hemicontinuity) For any $t \in [0, T]$ and $x, y, z \in \mathbb{X}$, the mapping

$$[0, 1] \ni \varepsilon \mapsto [x, A(t, y + \varepsilon z)]_{\mathbb{X}}$$

is continuous.

(H2) (Weak coercivity) There exist $q_1, q_2 \geq 2$ and $\lambda_1, \lambda_2, \lambda_3 > 0$ such that for all $x \in \mathbb{X}$ and $t \in [0, T]$

$$[x, A(t, x)]_{\mathbb{X}} \leq -\lambda_1 \cdot \|x\|_{\mathbb{X}_1}^{q_1} - \lambda_2 \cdot \|x\|_{\mathbb{X}_2}^{q_2} + \lambda_3 \cdot (\|x\|_{\mathbb{H}}^2 + 1).$$

(H3) (Weak monotonicity) There exist $\lambda_0, \lambda'_1, \lambda'_2 \geq 0$ such that for all $x, y \in \mathbb{X}$ and $t \in [0, T]$

$$\begin{aligned} [x - y, A(t, x) - A(t, y)]_{\mathbb{X}} &\leq -\lambda'_1 \|x - y\|_{\mathbb{X}_1}^{q_1} - \lambda'_2 \|x - y\|_{\mathbb{X}_2}^{q_2} \\ &\quad + \lambda_0 \cdot \|x - y\|_{\mathbb{H}}^2, \end{aligned}$$

where q_1 and q_2 are same as in **(H2)**.

(H4) (Boundedness) There exist $c_{A_1}, c_{A_2} > 0$ such that for all $x \in \mathbb{X}$ and $t \in [0, T]$

$$\|A_i(t, x)\|_{\mathbb{X}_i^*} \leq c_{A_i} \cdot (\|x\|_{\mathbb{X}_i}^{q_i-1} + 1), \quad i = 1, 2,$$

where q_1 and q_2 are same as in **(H2)**.

(H5) There exists a $\beta_1 > 0$ such that for all $x, y \in \mathbb{H}$ and $t \in [0, T]$

$$\|B(t, x) - B(t, y)\|_{L_2(\mathbb{U}_Q, \mathbb{H})} \leq \beta_1 \|x - y\|_{\mathbb{H}}$$

and

$$\|B(t, x)\|_{L_2(\mathbb{U}_Q, \mathbb{H})} \leq \beta_1 (1 + \|x\|_{\mathbb{H}}).$$

We take the polish space \mathbb{S} in Theorem 3.7 as follows

$$\mathbb{S} := \mathbb{C}_T(\mathbb{H}) \cap L^{q_1}(0, T; \mathbb{X}_1) \cap L^{q_2}(0, T; \mathbb{X}_2) \quad (5)$$

with the norm

$$\|x\|_{\mathbb{S}} := \sup_{t \in [0, T]} \|x(t)\|_{\mathbb{H}} + \sum_{i=1,2} \left(\int_0^T \|x(t)\|_{\mathbb{X}_i}^{q_i} dt \right)^{1/q_i}.$$

Consider the following stochastic evolution equation:

$$\begin{cases} dX(t) = A(t, X(t))dt + B(t, X(t))dW(t), \\ X(0) = x_0 \in \mathbb{H}. \end{cases} \quad (6)$$

By [18, 31] and [25], we have the following existence of unique strong solution to Eq.(6).

Theorem 2.4. *Assume that (H1)-(H5) hold. Then there exists a unique measurable functional Φ from $\mathbb{C}_T(\mathbb{U})$ to \mathbb{S} such that $X(t, \omega) = \Phi(W(\omega))(t)$ solves the following equation in \mathbb{X}^**

$$X(t) = x_0 + \int_0^t A(s, X(s))ds + \int_0^t B(s, X(s))dW(s),$$

where the Itô stochastic integral is calculated in Hilbert space \mathbb{H} . Moreover, for any $h \in \mathcal{A}_N$, $X^h(t, \omega) = \Phi(W(\omega) + h(\omega))(t)$ solves the following equation in \mathbb{X}^*

$$X(t) = x_0 + \int_0^t A(s, X(s))ds + \int_0^t B(s, X(s))dW(s) + \int_0^t B(s, X(s))\dot{h}(s)ds.$$

Remark 2.5. *The second conclusion follows from the Girsanov theorem.*

3. LAPLACE AND LARGE DEVIATION PRINCIPLE

Consider the following small perturbation to stochastic evolution equation (6):

$$\begin{cases} dX_\varepsilon(t) = A(t, X_\varepsilon(t))dt + \sqrt{\varepsilon}B(t, X_\varepsilon(t))dW(t), \quad \varepsilon \in (0, 1), \\ X_\varepsilon(0) = x_0 \in \mathbb{H}. \end{cases} \quad (7)$$

By Theorem 2.4, there exists a measurable mapping $\Phi_\varepsilon : \mathbb{C}_T(\mathbb{U}) \rightarrow \mathbb{S}$ such that

$$X_\varepsilon(t, \omega) = \Phi_\varepsilon(W(\omega))(t).$$

We now fix a family of processes $\{h^\varepsilon\}$ in \mathcal{A}_N , and put

$$X^\varepsilon(t, \omega) := \Phi_\varepsilon(W(\omega) + \frac{h^\varepsilon(\omega)}{\sqrt{\varepsilon}})(t).$$

It should be noticed that we have used a little confused notations X_ε and X^ε , but it is clearly different. Note that $X^\varepsilon(t)$ solves the following stochastic evolution equation:

$$\begin{cases} dX^\varepsilon(t) = A(t, X^\varepsilon(t))dt + \sqrt{\varepsilon}B(t, X^\varepsilon(t))dW(t) + B(t, X^\varepsilon(t))\dot{h}^\varepsilon(t)dt, \\ X^\varepsilon(0) = x_0 \in \mathbb{H}. \end{cases} \quad (8)$$

Moreover, the following energy identity holds(cf. [18], also called Itô's formula):

$$\begin{aligned} \|X^\varepsilon(t)\|_{\mathbb{H}}^2 &= \|x_0\|_{\mathbb{H}}^2 + 2 \int_0^t [X^\varepsilon(s), A(s, X^\varepsilon(s))]_{\mathbb{X}} ds + M^\varepsilon(t) \\ &\quad + 2 \int_0^t \langle X^\varepsilon(s), B(s, X^\varepsilon(s))\dot{h}^\varepsilon(s) \rangle_{\mathbb{H}} ds \\ &\quad + \varepsilon \int_0^t \|B(s, X^\varepsilon(s))\|_{L_2(\mathbb{U}_Q, \mathbb{H})}^2 ds, \end{aligned} \quad (9)$$

where $t \mapsto M^\varepsilon(t)$ is a real continuous martingale given by

$$M^\varepsilon(t) := 2\sqrt{\varepsilon} \int_0^t \langle X^\varepsilon(s), B(s, X^\varepsilon(s)) dW(s) \rangle_{\mathbb{H}}.$$

Note that the square variation process of $M^\varepsilon(t)$ is given by

$$\langle M^\varepsilon \rangle_t = 4\varepsilon \sum_j \int_0^t \langle X^\varepsilon(s), B(s, X^\varepsilon(s)) Q^{1/2}(e_j) \rangle_{\mathbb{H}}^2 ds,$$

where $\{e_j\}$ is an orthonormal basis of \mathbb{U} .

CONVENTION: The letter C below with or without subscripts will denote positive constants whose values may change in different occasions.

Our main task is to verify the above **(Hypothesis)**. We first prove some uniform estimates about $X^\varepsilon(t)$.

Lemma 3.1. *For any $p \geq 2$ and $T > 0$, there exists a constant $C_{p,T} > 0$ such that for all $\varepsilon \in (0, 1]$*

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|X^\varepsilon(t)\|_{\mathbb{H}}^{2p} \right) \leq C_{p,T} (\|x_0\|_{\mathbb{H}}^{2p} + 1),$$

and for $i = 1, 2$

$$\mathbb{E} \left(\int_0^T \|X^\varepsilon(s)\|_{\mathbb{X}_i}^{q_i} ds \right)^p \leq C_{p,T} (\|x_0\|_{\mathbb{H}}^{2p} + 1).$$

Proof. By (9) and Itô's formula, we find that

$$\begin{aligned} \|X^\varepsilon(t)\|_{\mathbb{H}}^{2p} &= \|x_0\|_{\mathbb{H}}^{2p} + 2p \int_0^t \|X^\varepsilon(s)\|_{\mathbb{H}}^{2p-2} \cdot [X^\varepsilon(s), A(s, X^\varepsilon(s))]_{\mathbb{X}} ds \\ &\quad + p \int_0^t \|X^\varepsilon(s)\|_{\mathbb{H}}^{2p-2} dM^\varepsilon(s) + \frac{p(p-1)}{2} \int_0^t \|X^\varepsilon(s)\|_{\mathbb{H}}^{2(p-2)} d\langle M^\varepsilon \rangle_s \\ &\quad + 2p \int_0^t \|X^\varepsilon(s)\|_{\mathbb{H}}^{2p-2} \langle X^\varepsilon(s), B(s, X^\varepsilon(s)) \dot{h}^\varepsilon(s) \rangle_{\mathbb{H}} ds \\ &\quad + \varepsilon p \int_0^t \|X^\varepsilon(s)\|_{\mathbb{H}}^{2p-2} \|B(s, X^\varepsilon(s))\|_{L_2(\mathbb{U}_Q, \mathbb{H})}^2 ds. \end{aligned}$$

By **(H2)** and **(H5)** we have

$$\begin{aligned} \|X^\varepsilon(t)\|_{\mathbb{H}}^{2p} &\leq \|x_0\|_{\mathbb{H}}^{2p} + C \int_0^t \left(\|X^\varepsilon(s)\|_{\mathbb{H}}^{2p} (\|\dot{h}^\varepsilon(s)\|_{\mathbb{U}_Q} + 1) + 1 \right) ds \\ &\quad + p \int_0^t \|X^\varepsilon(s)\|_{\mathbb{H}}^{2p-2} dM^\varepsilon(s). \end{aligned}$$

Hence, by Gronwall's inequality and (2) we get

$$\|X^\varepsilon(t)\|_{\mathbb{H}}^{2p} \leq C_N \left(\|x_0\|_{\mathbb{H}}^{2p} + 1 + \int_0^t \left| \int_0^s \|X^\varepsilon(r)\|_{\mathbb{H}}^{2p-2} dM^\varepsilon(r) \right| ds \right).$$

Put

$$f(t) := \mathbb{E} \left(\sup_{r \in [0, t]} \|X^\varepsilon(r)\|_{\mathbb{H}}^{2p} \right).$$

Then, by BDG's inequality and Young's inequality we obtain

$$f(t) \leq C_N \cdot (\|x_0\|_{\mathbb{H}}^{2p} + 1) + C_N \cdot T \cdot \mathbb{E} \left(\int_0^t \|X^\varepsilon(r)\|_{\mathbb{H}}^{4p-4} d\langle M^\varepsilon \rangle_r \right)^{1/2}$$

$$\begin{aligned}
&\leq C_N \cdot (\|x_0\|_{\mathbb{H}}^{2p} + 1) + C_N \mathbb{E} \left(\sup_{r \in [0, t]} \|X^\varepsilon(r)\|_{\mathbb{H}}^{2p} \cdot \int_0^t (\|X^\varepsilon(r)\|_{\mathbb{H}}^{2p} + 1) dr \right)^{1/2} \\
&\leq C_N \cdot (\|x_0\|_{\mathbb{H}}^{2p} + 1) + \frac{1}{2} f(t) + C_N \int_0^t (\mathbb{E} \|X^\varepsilon(r)\|_{\mathbb{H}}^{2p} + 1) dr.
\end{aligned}$$

Therefore,

$$f(t) \leq 2C_N \cdot (\|x_0\|_{\mathbb{H}}^{2p} + 1) + 2C_N \int_0^t (f(r) + 1) dr.$$

By Gronwall's inequality again, we obtain the first estimate.

As for the second estimate, from (9) and **(H2)**, **(H5)** we also have

$$\sum_{i=1,2} \int_0^T \|X^\varepsilon(s)\|_{\mathbb{X}_i}^{q_i} ds \leq C \left(\|x_0\|_{\mathbb{H}}^2 + |M^\varepsilon(t)| + \int_0^t \|X^\varepsilon(s)\|_{\mathbb{H}}^2 (\|\dot{h}^\varepsilon(s)\|_{\mathbb{U}_Q} + 1) ds \right).$$

Using the first estimate, we immediately get the desired second estimate. \square

Lemma 3.2. *For any $p \geq 2$, there exists a constant C depending on p, T, N and x_0 such that for all $t, r \in [0, T]$ and $\varepsilon \in (0, 1)$*

$$\mathbb{E} \|X^\varepsilon(t) - X^\varepsilon(r)\|_{\mathbb{X}^*}^p \leq C |t - r|^{\frac{p}{q_1 \vee q_2}}.$$

Proof. Note that the following equality holds in \mathbb{X}^*

$$\begin{aligned}
X^\varepsilon(t) - X^\varepsilon(r) &= \int_r^t A(s, X^\varepsilon(s)) ds + \sqrt{\varepsilon} \int_r^t B(s, X^\varepsilon(s)) dW(s) \\
&\quad + \int_r^t B(s, X^\varepsilon(s)) \dot{h}^\varepsilon(s) ds.
\end{aligned}$$

Hence

$$\mathbb{E} \|X^\varepsilon(t) - X^\varepsilon(r)\|_{\mathbb{X}^*}^p \leq 3^{p-1} (I_1 + I_2 + I_3),$$

where

$$\begin{aligned}
I_1 &:= \mathbb{E} \left(\int_r^t \|A(s, X^\varepsilon(s))\|_{\mathbb{X}^*} ds \right)^p \\
I_2 &:= \mathbb{E} \left\| \sqrt{\varepsilon} \int_r^t B(s, X^\varepsilon(s)) dW(s) \right\|_{\mathbb{X}^*}^p \\
I_3 &:= \mathbb{E} \left\| \int_r^t B(s, X^\varepsilon(s)) \dot{h}^\varepsilon(s) ds \right\|_{\mathbb{X}^*}^p.
\end{aligned}$$

For I_1 , we have by **(H4)** and Hölder's inequality

$$\begin{aligned}
I_1 &\leq C \mathbb{E} \left(\int_r^t (\|A_1(s, X^\varepsilon(s))\|_{\mathbb{X}_1^*} + \|A_2(s, X^\varepsilon(s))\|_{\mathbb{X}_2^*}) ds \right)^p \\
&\leq C \mathbb{E} \left(\int_r^t (\|X^\varepsilon(s)\|_{\mathbb{X}_1}^{q_1-1} + \|X^\varepsilon(s)\|_{\mathbb{X}_2}^{q_2-1} + 1) ds \right)^p \\
&\leq C(t-r)^p + C \sum_{i=1,2} \left[\mathbb{E} \left(\int_r^t \|X^\varepsilon(s)\|_{\mathbb{X}_i}^{q_i} ds \right)^{\frac{p(q_i-1)}{q_i}} (t-r)^{\frac{p}{q_i}} \right].
\end{aligned}$$

For I_2 , we have by BDG's inequality and **(H5)**

$$I_2 \leq C \mathbb{E} \left\| \sqrt{\varepsilon} \int_r^t B(s, X^\varepsilon(s)) dW(s) \right\|_{\mathbb{H}}^p$$

$$\begin{aligned}
&\leq C\mathbb{E} \left(\int_r^t \|B(s, X^\varepsilon(s))\|_{L_2(\mathbb{U}_Q, \mathbb{H})}^2 ds \right)^{p/2} \\
&\leq C|t-r|^{p/2-1} \left(\int_r^t (\mathbb{E}\|X^\varepsilon(s)\|_{\mathbb{H}}^p + 1) ds \right).
\end{aligned}$$

For I_3 , we have by Hölder's inequality, (2) and **(H5)**

$$\begin{aligned}
I_3 &\leq C\mathbb{E} \left\| \int_r^t B(s, X^\varepsilon(s)) \dot{h}^\varepsilon(s) ds \right\|_{\mathbb{H}}^p \\
&\leq C\mathbb{E} \left(\int_r^t \|B(s, X^\varepsilon(s))\|_{L_2(\mathbb{U}_Q, \mathbb{H})} \cdot \|\dot{h}^\varepsilon(s)\|_{\mathbb{U}_Q} ds \right)^p \\
&\leq C\mathbb{E} \left(\int_r^t \|B(s, X^\varepsilon(s))\|_{L_2(\mathbb{U}_Q, \mathbb{H})}^2 ds \right)^{p/2} \left(\int_0^T \|\dot{h}^\varepsilon(s)\|_{\mathbb{U}_Q}^2 ds \right)^{p/2} \\
&\leq C|t-r|^{p/2-1} \left(\int_r^t (\mathbb{E}\|X^\varepsilon(s)\|_{\mathbb{H}}^p + 1) ds \right) \cdot N^{p/2}.
\end{aligned} \tag{10}$$

The desired estimate now follows by combining the above estimates and Lemma 3.1. \square

Lemma 3.3. *Assume that for almost all ω , $h^\varepsilon(\cdot, \omega)$ weakly converge to $h(\cdot, \omega)$ in \mathbb{L}_Q , and $X^\varepsilon(\cdot, \omega)$ strongly converge to $X(\cdot, \omega)$ in $\mathbb{C}_T(\mathbb{X}^*)$. Then $X(\cdot, \omega)$ solves the following equation*

$$X(t, \omega) = x_0 + \int_0^t A(s, X(s, \omega)) ds + \int_0^t B(s, X(s, \omega)) \dot{h}(s, \omega) ds.$$

Moreover, there exists a subsequence ε_k such that as $k \rightarrow \infty$,

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|X^{\varepsilon_k}(t) - X(t)\|_{\mathbb{H}}^2 \right) \rightarrow 0, \tag{11}$$

and if $\lambda'_1, \lambda'_2 > 0$ in **(H3)**, then for $i = 1, 2$

$$\int_0^T \mathbb{E} \|X^{\varepsilon_k}(t) - X(t)\|_{\mathbb{X}_i}^{q_i} dt \rightarrow 0. \tag{12}$$

Proof. Set for $i = 1, 2$

$$\mathbb{K}_{1,i} := L^{q_i/(q_i-1)}([0, T] \times \Omega, \mathcal{M}, dt \times dP; \mathbb{X}_i^*)$$

and

$$\mathbb{K}_{2,i} := L^{q_i}([0, T] \times \Omega, \mathcal{M}, dt \times dP; \mathbb{X}_i),$$

where \mathcal{M} denotes the progressively σ -algebra associated with \mathcal{F}_t . Then $\mathbb{K}_{1,i}$ and $\mathbb{K}_{2,i}$ are reflexive and separable Banach spaces.

We have by Lemma 3.1

$$\sup_{\varepsilon \in (0, 1]} \mathbb{E} \|X^\varepsilon(T)\|_{\mathbb{H}}^2 + \sup_{\varepsilon \in (0, 1]} \sum_{i=1,2} \|X^\varepsilon\|_{\mathbb{K}_{2,i}}^{q_i} < +\infty \tag{13}$$

and

$$\sup_{\varepsilon \in (0, 1]} \mathbb{E} \left(\sup_{t \in [0, T]} \|X^\varepsilon(t)\|_{\mathbb{H}}^4 \right) < +\infty. \tag{14}$$

Hence, by the strong convergence of $X^\varepsilon(\cdot, \omega)$ to $X(\cdot, \omega)$ in $\mathbb{C}_T(\mathbb{X}^*)$ we have

$$\mathbb{E} \|X(T)\|_{\mathbb{H}}^2 \leq \liminf_{\varepsilon \downarrow 0} \mathbb{E} \|X^\varepsilon(T)\|_{\mathbb{H}}^2 \leq C_{p,T} (\|x_0\|_{\mathbb{H}}^2 + 1) \tag{15}$$

$$\int_0^T \mathbb{E} \|X(s)\|_{\mathbb{X}_i}^{q_i} ds \leq \lim_{\varepsilon \downarrow 0} \int_0^T \mathbb{E} \|X^\varepsilon(s)\|_{\mathbb{X}_i}^{q_i} ds \leq C_{p,T}(\|x_0\|_{\mathbb{H}}^2 + 1), \quad (16)$$

as well as by (14)

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left(\sup_{t \in [0, T]} \|X^\varepsilon(s) - X(s)\|_{\mathbb{X}^*}^2 \right) = 0.$$

Thus, by (1) we have as $\varepsilon \downarrow 0$

$$\begin{aligned} \mathbb{E} \left(\int_0^T \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}}^2 ds \right) &= \mathbb{E} \left(\int_0^T [X^\varepsilon(s) - X(s), X^\varepsilon(s) - X(s)]_{\mathbb{X}} ds \right) \\ &\leq \int_0^T \mathbb{E} \left(\|X^\varepsilon(s) - X(s)\|_{\mathbb{X}} \cdot \|X^\varepsilon(s) - X(s)\|_{\mathbb{X}^*} \right) ds \\ &\leq \left(\int_0^T \mathbb{E} \|X^\varepsilon(s) - X(s)\|_{\mathbb{X}}^2 ds \cdot \int_0^T \mathbb{E} \|X^\varepsilon(s) - X(s)\|_{\mathbb{X}^*}^2 ds \right)^{1/2} \rightarrow 0. \end{aligned} \quad (17)$$

Notice also that by **(H4)** and (13)

$$\sup_{\varepsilon \in (0, 1]} \|A_i(\cdot, X^\varepsilon(\cdot))\|_{\mathbb{K}_{1,i}} < +\infty, \quad i = 1, 2.$$

By this and (13) and the weak compactness of $\mathbb{K}_{1,i}$ and $\mathbb{K}_{2,i}$, $i = 1, 2$, there exist a subsequence ε_k (still denoted by ε for simplicity) and $Y_i \in \mathbb{K}_{1,i}$, $i = 1, 2$, $\bar{X} \in \cap_{i=1,2} \mathbb{K}_{2,i}$, $X_T \in L^2(\Omega)$ such that

$$X^\varepsilon \rightarrow \bar{X} \text{ weakly in } \mathbb{K}_{2,i}, i = 1, 2, \quad (18)$$

$$X^\varepsilon(T) \rightarrow X_T \text{ weakly in } L^2(\Omega), \quad (19)$$

and

$$Y_i^\varepsilon := A_i(\cdot, X^\varepsilon(\cdot)) \rightarrow Y_i \text{ weakly in } \mathbb{K}_{1,i}, i = 1, 2. \quad (20)$$

Put $Y = Y_1 + Y_2$ and define

$$\tilde{X}(t) := x_0 + \int_0^t Y(s) ds + \int_0^t B(s, X(s)) \dot{h}(s) ds.$$

Note that

$$\begin{aligned} X^\varepsilon(t) &= x_0 + \int_0^t A(s, X^\varepsilon(s)) ds + \sqrt{\varepsilon} \int_0^t B(s, X^\varepsilon(s)) dW^\varepsilon(s) \\ &\quad + \int_0^t B(s, X^\varepsilon(s)) \dot{h}^\varepsilon(s) ds. \end{aligned}$$

By taking weak limits and (17), it is not hard to see that (see also the proof of (25) below)

$$\tilde{X}(t, \omega) = X(t, \omega) = \bar{X}(t, \omega) \text{ for } dt \times dP\text{-almost all } (t, \omega)$$

and

$$\tilde{X}(T) = X(T) = X_T. \quad (21)$$

In the following we use the unified notation X , and only need to prove by the usual monotonicity argument that

$$Y(s, \omega) = A(s, X(s, \omega)) \text{ for } dt \times dP\text{-almost all } (t, \omega). \quad (22)$$

Without loss of generality, we assume that $\lambda_0 = 0$ in **(H3)** (cf. [21, 31]). It is clear that in (9)

$$\mathbb{E} M^\varepsilon(T) = 0, \quad (23)$$

and

$$\lim_{\varepsilon \downarrow 0} \varepsilon \mathbb{E} \int_0^T \|B(s, X^\varepsilon(s))\|_{L_2(\mathbb{U}_Q, \mathbb{H})}^2 ds = 0. \quad (24)$$

Let us prove the following limit:

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left| \int_0^T \left(\langle X^\varepsilon(s), B(s, X^\varepsilon(s)) \dot{h}^\varepsilon(s) \rangle_{\mathbb{H}} - \langle X(s), B(s, X(s)) \dot{h}(s) \rangle_{\mathbb{H}} \right) ds \right| = 0. \quad (25)$$

Since for almost all ω , $h^\varepsilon(\cdot, \omega)$ weakly converges to $h(\cdot, \omega)$ in \mathbb{L}_Q , by the dominated convergence theorem we have

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left| \int_0^T \langle X(s), B(s, X(s)) (\dot{h}^\varepsilon(s) - \dot{h}(s)) \rangle_{\mathbb{H}} ds \right| = 0.$$

By (2), Lemma 3.1 and (17) we also have

$$\begin{aligned} & \mathbb{E} \left| \int_0^T \langle X^\varepsilon(s) - X(s), B(s, X^\varepsilon(s)) \dot{h}^\varepsilon(s) \rangle_{\mathbb{H}} ds \right| \\ & \leq \mathbb{E} \int_0^T \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}} \cdot (\|X^\varepsilon(s)\|_{\mathbb{H}} + 1) \cdot \|\dot{h}^\varepsilon(s)\|_{\mathbb{U}_Q} ds \\ & \leq C_N \mathbb{E} \left(\int_0^T \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}}^2 \cdot (\|X^\varepsilon(s)\|_{\mathbb{H}} + 1)^2 ds \right)^{1/2} \\ & \leq C_N \mathbb{E} \left(\left(\sup_{s \in [0, T]} \|X^\varepsilon(s)\|_{\mathbb{H}}^2 + 1 \right) \cdot \int_0^T \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}}^2 \cdot ds \right)^{1/2} \\ & \leq C_N \left(\int_0^T \mathbb{E} \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}}^2 ds \right)^{1/2} \rightarrow 0, \end{aligned} \quad (26)$$

and

$$\begin{aligned} & \mathbb{E} \left| \int_0^T \langle X(s), (B(s, X^\varepsilon(s)) - B(s, X(s))) \dot{h}^\varepsilon(s) \rangle_{\mathbb{H}} ds \right| \\ & \leq \mathbb{E} \int_0^T \|X(s)\|_{\mathbb{H}} \cdot \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}} \cdot \|\dot{h}^\varepsilon(s)\|_{\mathbb{U}_Q} ds \\ & \leq C_N \mathbb{E} \left(\int_0^T \|X(s)\|_{\mathbb{H}}^2 \cdot \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}}^2 ds \right)^{1/2} \rightarrow 0. \end{aligned}$$

The limit (25) now follows.

Notice that for any $\Phi \in \mathbb{K}_{2,1} \cap \mathbb{K}_{2,2}$

$$\begin{aligned} \mathbb{E} \int_0^T [X^\varepsilon(s), A(s, X^\varepsilon(s))]_{\mathbb{X}} ds & \leq \mathbb{E} \int_0^T [\Phi(s), A(s, X^\varepsilon(s)) - A(s, \Phi(s))]_{\mathbb{X}} ds \\ & \quad + \mathbb{E} \int_0^T [X^\varepsilon(s), A(s, \Phi(s))]_{\mathbb{X}} ds \quad (\because \lambda_0 = 0) \\ & \rightarrow \mathbb{E} \int_0^T [\Phi(s), Y(s) - A(s, \Phi(s))]_{\mathbb{X}} ds \\ & \quad + \mathbb{E} \int_0^T [X(s), A(s, \Phi(s))]_{\mathbb{X}} ds, \end{aligned} \quad (27)$$

as $\varepsilon \downarrow 0$. The limits are due to (18) and (20).

Combining (9), (15), (23)-(27) yields that

$$\begin{aligned}
& \mathbb{E}\|X_T\|_{\mathbb{H}}^2 \leq \lim_{\varepsilon \downarrow 0} \mathbb{E}\|X^\varepsilon(T)\|_{\mathbb{H}}^2 \\
& \leq \|x_0\|_{\mathbb{H}}^2 + 2\mathbb{E} \int_0^T [\Phi(s), Y(s) - A(s, \Phi(s))]_{\mathbb{X}} ds \\
& \quad + 2\mathbb{E} \int_0^T [X(s), A(s, \Phi(s))]_{\mathbb{X}} ds \\
& \quad + 2\mathbb{E} \int_0^T \langle X(s), B(s, X(s)) \dot{h}(s) \rangle_{\mathbb{H}} ds.
\end{aligned}$$

On the other hand, by the energy equality (see (9)) we have

$$\|\tilde{X}(T)\|_{\mathbb{H}}^2 = \|x_0\|_{\mathbb{H}}^2 + 2 \int_0^T [X(s), Y(s)]_{\mathbb{X}} ds + 2 \int_0^T \langle X(s), B(s, X(s)) \dot{h}(s) \rangle_{\mathbb{H}} ds.$$

So by (21)

$$\mathbb{E} \int_0^T [X(s) - \Phi(s), Y(s) - A(s, \Phi(s))]_{\mathbb{X}} ds \leq 0,$$

which then yields (22) by **(H1)** and [31, Lemma 2.5] (see also [21]).

Lastly, let us prove the limits (11) and (12). By Itô's formula, we have

$$\|X^\varepsilon(t) - X(t)\|_{\mathbb{H}}^2 = I_1^\varepsilon(t) + I_2^\varepsilon(t) + I_3^\varepsilon(t) + I_4^\varepsilon(t).$$

where

$$\begin{aligned}
I_1^\varepsilon(t) &:= 2 \int_0^t [X^\varepsilon(s) - X(s), A(s, X^\varepsilon(s)) - A(s, X(s))]_{\mathbb{X}} ds \\
I_2^\varepsilon(t) &:= 2 \int_0^t \langle X^\varepsilon(s) - X(s), B(s, X^\varepsilon(s)) \dot{h}^\varepsilon(s) \rangle_{\mathbb{H}} ds \\
I_3^\varepsilon(t) &:= -2 \int_0^t \langle X^\varepsilon(s) - X(s), B(s, X(s)) \dot{h}(s) \rangle_{\mathbb{H}} ds \\
I_4^\varepsilon(t) &:= 2\sqrt{\varepsilon} \int_0^t \langle X^\varepsilon(s) - X(s), B(s, X^\varepsilon(s)) dW(s) \rangle_{\mathbb{H}} \\
I_5^\varepsilon(t) &:= \varepsilon \int_0^t \|B(s, X^\varepsilon(s))\|_{L_2(\mathbb{U}_Q, \mathbb{H})}^2 ds.
\end{aligned}$$

By BDG's inequality and Lemma 3.1, we obviously have

$$\lim_{k \rightarrow \infty} \mathbb{E} \left(\sup_{t \in [0, T]} (|I_4^\varepsilon(t)| + |I_5^\varepsilon(t)|) \right) = 0.$$

For I_2^ε , as in the proof of (26) we have

$$\begin{aligned}
\mathbb{E} \left(\sup_{t \in [0, T]} |I_2^\varepsilon(t)| \right) &\leq C \mathbb{E} \int_0^T \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}} \cdot (\|X^\varepsilon(s)\|_{\mathbb{H}} + 1) \cdot \|\dot{h}^\varepsilon(s)\|_{\mathbb{U}_Q} ds \\
&\leq C_N \mathbb{E} \left(\int_0^T \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}}^2 \cdot (\|X^\varepsilon(s)\|_{\mathbb{H}} + 1)^2 ds \right)^{1/2} \\
&\rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned}$$

Similarly

$$\lim_{k \rightarrow \infty} \mathbb{E} \left(\sup_{t \in [0, T]} |I_3^\varepsilon(t)| \right) = 0.$$

Assume $\lambda'_1, \lambda'_2 > 0$, then

$$I_1^\varepsilon(t) \leq - \sum_{i=1,2} \lambda'_i \int_0^t \|X^\varepsilon(s) - X(s)\|_{\mathbb{X}_i}^{q_i} ds + \lambda_0 \int_0^t \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}}^2 ds. \quad (28)$$

If we put

$$f(t) := \overline{\lim}_{\varepsilon \rightarrow \infty} \mathbb{E} \left(\sup_{s \in [0, t]} \|X^\varepsilon(s) - X(s)\|_{\mathbb{H}}^2 \right),$$

then

$$f(t) \leq \lambda_0 \int_0^t f(s) ds = 0.$$

So

$$f(T) = 0.$$

The limits (11) and (12) is straightforward by noting (28). \square

We may prove the following main lemma.

Lemma 3.4. *There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and a sequence (still indexed by ε for simplicity) $\{(\tilde{h}^\varepsilon, \tilde{X}^\varepsilon, \tilde{W}^\varepsilon)\}$ and (h, X^h, \tilde{W}) defined on this probability space and taking values in $D_N \times \mathbb{C}_T(\mathbb{X}^*) \times \mathbb{C}_T(\mathbb{U})$ such that*

- (a) $(\tilde{h}^\varepsilon, \tilde{X}^\varepsilon, \tilde{W}^\varepsilon)$ has the same law as $(h^\varepsilon, X^\varepsilon, W)$ for each ε ;
- (b) $(\tilde{h}^\varepsilon, \tilde{X}^\varepsilon, \tilde{W}^\varepsilon) \rightarrow (h, X^h, \tilde{W})$ in $D_N \times \mathbb{C}_T(\mathbb{X}^*) \times \mathbb{C}_T(\mathbb{U})$, \tilde{P} -a.s. as $\varepsilon \rightarrow 0$;
- (c) (h, X^h) uniquely solves the following equation:

$$X^h(t) = x_0 + \int_0^t A(s, X^h(s)) ds + \int_0^t B(s, X^h(s)) \dot{h}(s) ds. \quad (29)$$

Moreover, there exists a subsequence ε_k such that as $k \rightarrow \infty$,

$$\mathbb{E}^{\tilde{P}} \left(\sup_{t \in [0, T]} \|\tilde{X}^{\varepsilon_k}(t) - X^h(t)\|_{\mathbb{H}}^2 \right) \rightarrow 0, \quad (30)$$

and if $\lambda'_1, \lambda'_2 > 0$ in **(H3)**, then for $i = 1, 2$

$$\int_0^T \mathbb{E}^{\tilde{P}} \|\tilde{X}^{\varepsilon_k}(t) - X^h(t)\|_{\mathbb{X}_i}^{q_i} dt \rightarrow 0. \quad (31)$$

Proof. By Lemma 3.2 and [17, Corollary 14.9], the laws of $(h^\varepsilon, X^\varepsilon, W)$ in $D_N \times \mathbb{C}_T(\mathbb{X}^*) \times \mathbb{C}_T(\mathbb{U})$ is tight. By Skorohod's embedding theorem, the conclusions (a) and (b) hold. Note that $\tilde{X}^\varepsilon(0) = x_0$ \tilde{P} -a.s. and

$$\begin{aligned} \tilde{X}^\varepsilon(t) &= x_0 + \int_0^t A(s, \tilde{X}^\varepsilon(s)) ds + \sqrt{\varepsilon} \int_0^t B(s, \tilde{X}^\varepsilon(s)) d\tilde{W}^\varepsilon(s) \\ &\quad + \int_0^t B(s, \tilde{X}^\varepsilon(s)) \dot{\tilde{h}}^\varepsilon(s) ds. \end{aligned}$$

The other conclusions follows from Lemma 3.3. \square

From this lemma, one sees that **(Hypothesis)** holds. Thus, by Theorem 3.7 we obtain

Theorem 3.5. Assume (H1)-(H5) hold, and $\lambda'_1, \lambda'_2 > 0$ in (H3). Then for all real bounded continuous functions g on \mathbb{S}

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left(\exp \left[-\frac{g(X_\varepsilon)}{\varepsilon} \right] \right) = -\inf_{f \in \mathbb{S}} \{g(f) + I(f)\},$$

where $I(f)$ is defined by

$$I(f) := \frac{1}{2} \inf_{\{h \in \mathbb{L}_Q : f = X^h\}} \|h\|_{\mathbb{L}_Q}^2, \quad (32)$$

and X^h solves (29).

Remark 3.6. If $\lambda'_1, \lambda'_2 = 0$ in (H3), then the conclusion still holds if \mathbb{S} is replaced by $\mathbb{C}_T(\mathbb{H})$.

In order to show the large deviation principle, we need to prove that $I(f)$ is a good rate function. For this aim, we need an extra assumption:

$$\mathbb{X} \hookrightarrow \mathbb{H} \text{ compactly.}$$

Lemma 3.7. In addition to (H1)-(H5) and $\lambda'_1, \lambda'_2 > 0$, we also assume that \mathbb{X} is compactly embedded in \mathbb{H} . Then $I(f)$ is a good rate function, i.e., for any $a > 0$, $\{f \in \mathbb{S} : I(f) \leq a\}$ is compact.

Proof. It suffices to prove that if $h_n \in D_N$ weakly converge to h in \mathbb{L}_Q , then there exists a subsequence n_k (still denoted by n) such that

$$\lim_{n \rightarrow \infty} \|X^{h_n} - X^h\|_{\mathbb{S}} = 0. \quad (33)$$

In fact, assume that $I(f_n) \leq a$. By the definition of $I(f_n)$, there exists a sequence $h_n \in \mathbb{L}_Q$ such that $X^{h_n} = f_n$ and

$$\frac{1}{2} \|h_n\|_{\mathbb{L}_Q}^2 \leq a + \frac{1}{n}.$$

By the weak compactness of D_{2a+1} , there exists a subsequence n_k (still denoted by n) and $h \in \mathbb{L}_Q$ such that h_n weakly converge to h and

$$\|h\|_{\mathbb{L}_Q}^2 \leq \liminf_{n \rightarrow \infty} \|h_n\|_{\mathbb{L}_Q}^2 \leq 2a.$$

Thus, by (33) we get the desired compactness.

We now prove (33). As in the proofs of Lemma 3.1 and Lemma 3.2, we may prove

$$\|X^{h_n}\|_{\mathbb{S}} = \sup_{t \in [0, T]} \|X^{h_n}(t)\|_{\mathbb{H}} + \sum_{i=1,2} \left(\int_0^T \|X^{h_n}(t)\|_{\mathbb{X}_i}^{q_i} dt \right)^{1/q_i} \leq C$$

and

$$\|X^{h_n}(t) - X^{h_n}(r)\|_{\mathbb{X}^*} \leq C|t - r|^{\frac{1}{q_1 \vee q_2}},$$

where C is independent of n .

Since $\mathbb{X} \hookrightarrow \mathbb{H}$ is compact, by [13, Theorem 2.1] there exists a subsequence n_k (still denoted by n) and an $X \in L^2(0, T; \mathbb{H})$ such that

$$\int_0^T \|X^{h_n}(t) - X(t)\|_{\mathbb{H}}^2 dt = 0.$$

Basing on this convergence, as in the proof of Lemma 3.3, we in fact have $X = X^h$ and the desired limit (33) hold. \square

Using Theorem 3.5 and Lemma 3.7, we obtain the following large deviation principle.

Theorem 3.8. Assume (H1)-(H5) hold, and \mathbb{X} is compactly embedded in \mathbb{H} , $\lambda'_1, \lambda'_2 > 0$ in (H3). Let the law of X_ε in \mathbb{S} be denoted by ν_ε . Then for any $A \in \mathcal{B}(\mathbb{S})$

$$-\inf_{f \in A^o} I(f) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon \log \nu_\varepsilon(A) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \nu_\varepsilon(A) \leq -\inf_{f \in \bar{A}} I(f),$$

where the closure and the interior are taken in \mathbb{S} , and $I(f)$ is a good rate function defined by (32).

4. APPLICATIONS

4.1. SDE with monotone drift. We consider the following small perturbation of stochastic ordinary differential equation with monotone drift:

$$dX_\varepsilon(t) = b(t, X_\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(t, X_\varepsilon(t))dW(t), \quad X(0) = x_0 \in \mathbb{R}^d,$$

where W is an m -dimensional Brownian motion, σ and b satisfy that

(H σ) There exists a constant $C_\sigma > 0$ such that for all $t \in [0, T]$ and $x \in \mathbb{R}^d$

$$\|\sigma(t, x) - \sigma(t, y)\|_{\mathbb{R}^{d \times m}} \leq C_\sigma \|x - y\|_{\mathbb{R}^d}.$$

(Hb) There exists a constant $C_b > 0$ such that for all $t \in [0, T]$ and $x \in \mathbb{R}^d$

$$\langle x - y, b(t, x) - b(t, y) \rangle_{\mathbb{R}^d} \leq C_b \|x - y\|_{\mathbb{R}^d}^2.$$

Let ν_ε be the law of $X_\varepsilon(t)$ in the continuous functions space $\mathbb{C}_T(\mathbb{R}^d)$. Then the conclusion of Theorem 3.8 holds.

The following two examples can also be found in [32].

4.2. Stochastic reaction diffusion equation. Let \mathcal{O} be an open and bounded set in Euclidean space \mathbb{R}^d , where the boundary $\partial\mathcal{O}$ of \mathcal{O} is assumed to be smooth. For $q \geq 2$, let $W_0^{1,q}(\mathcal{O})$ and $W^{-1, \frac{q}{q-1}}(\mathcal{O})$ be the usual Sobolev spaces(cf. [1]).

Suppose that for each integer $j = 1, \dots, d$, we are given a function $a_j : \mathcal{O} \times \mathbb{R} \mapsto \mathbb{R}$ such that

$$r \mapsto a_j(\xi, r) \text{ is continuous and non-decreasing for each } \xi \in \mathcal{O}, \quad (34)$$

$$a_j(\xi, r)\xi \geq C_1|r|^{q_1} - C_2, \quad (\xi, r) \in \mathcal{O} \times \mathbb{R}, \quad (35)$$

$$|a_j(\xi, r)| \leq C_3(|r|^{q_1-1} + 1), \quad (\xi, r) \in \mathcal{O} \times \mathbb{R}, \quad (36)$$

where $q_1 \geq 2$ and $C_1, C_2, C_3 > 0$.

Let b be another continuous function satisfying (34)-(36) and with a different constant $q_2 \geq 2$. Let l^2 be the usual Hilbert space of square summable real number sequences. Let $\sigma(\xi, r) : \mathcal{O} \times \mathbb{R} \mapsto l^2$ satisfy that $\sigma(\cdot, 0) \in L^2(\mathcal{O}; l^2)$ and for some $c_1 > 0$

$$\|\sigma(\xi, r) - \sigma(\xi, r')\|_{l^2} \leq c_1 \cdot |r - r'|, \quad \xi \in \mathcal{O}, r, r' \in \mathbb{R},$$

We consider the following small perturbation of stochastic reaction diffusion equation

$$\begin{cases} dX_\varepsilon(t, \xi) = \left[\sum_{i=1}^d \partial_i a_i(\xi, \partial_i X_\varepsilon(t, \xi)) - b(\xi, X_\varepsilon(t, \xi)) \right] dt \\ \quad + \sqrt{\varepsilon} \sum_{j=1}^\infty \sigma_j(\xi, X_\varepsilon(t, \xi)) dW_j(t), \\ X_\varepsilon(t, \xi) = 0, \quad \forall \xi \in \partial\mathcal{O}, \\ X_\varepsilon(0, \xi) = x(\xi) \in L^2(\mathcal{O}), \end{cases} \quad (37)$$

where $W_j(t) = \langle W(t), \ell_j \rangle_{\mathbb{U}_Q}$ and $\{\ell_j, j \in \mathbb{N}\}$ is an orthogonal basis of \mathbb{U}_Q .

Set

$$\mathbb{X}_1 := W_0^{1, q_1}(\mathcal{O}), \quad \mathbb{X}_2 := L^{q_2}(\mathcal{O}), \quad \mathbb{H} := L^2(\mathcal{O})$$

and

$$\mathbb{X}_1^* := W^{-1, \frac{q_1}{q_1-1}}(\mathcal{O}), \quad \mathbb{X}_2^* := L^{\frac{q_2}{q_2-1}}(\mathcal{O}).$$

Then

$$\mathbb{X} := \mathbb{X}_1 \cap \mathbb{X}_2 \subset \mathbb{H} \subset (\mathbb{X}_1^* + \mathbb{X}_2^*) \subset \mathbb{X}^*$$

forms an evolutionary triple.

Now, define for $u, v \in \mathbb{X}_1$

$$[A_1(u), v]_{\mathbb{X}_1} := - \sum_{i=1}^d \int_{\mathcal{O}} a_i(\xi, \partial_i u(\xi)) \cdot \partial_i v(\xi) d\xi$$

and for $u, v \in \mathbb{X}_2$

$$[A_2(u), v]_{\mathbb{X}_2} := - \int_{\mathcal{O}} b(\xi, u(\xi)) \cdot v(\xi) d\xi.$$

Clearly, for each $u \in \mathbb{X}_1$, $[A_1(u), \cdot]_{\mathbb{X}_1} \in \mathbb{X}_1^*$ and for each $u \in \mathbb{X}_2$, $[A_2(u), \cdot]_{\mathbb{X}_2} \in \mathbb{X}_2^*$. Thus,

$$A_1 : \mathbb{X}_1 \mapsto \mathbb{X}_1^*, \quad A_2 : \mathbb{X}_2 \mapsto \mathbb{X}_2^*,$$

and it is easy to verify that $A := A_1 + A_2$ satisfies **(H1)**-**(H4)**.

Moreover, if we define for $x \in \mathbb{H} = L^2(\mathcal{O})$

$$B(t, x) := \sum_{j=1}^{\infty} \sigma_j(\cdot, x(\cdot)) \ell_j$$

then for any $x, y \in L^2(\mathcal{O})$

$$\begin{aligned} \|B(t, x) - B(t, y)\|_{L^2(\mathbb{U}_Q; \mathbb{H})}^2 &= \sum_{j=1}^{\infty} \|\sigma_j(\cdot, x(\cdot)) - \sigma_j(\cdot, y(\cdot))\|_{L^2(\mathcal{O})}^2 \\ &\leq C \|x(\cdot) - y(\cdot)\|_{L^2(\mathcal{O})}^2. \end{aligned}$$

Thus, **(H5)** holds.

Let ν_ε be the law of $X_\varepsilon(t)$ in \mathbb{S} , where \mathbb{S} is defined by (5). Then the conclusion of Theorem 3.8 holds.

4.3. Stochastic Porous Medium Equation. As in the previous subsection, we consider the bounded domain \mathcal{O} in \mathbb{R}^d with smooth boundary.

For $p \geq 2$, set

$$\mathbb{X} := L^p(\mathcal{O}), \quad \mathbb{H} := W^{-1,2}(\mathcal{O}), \quad \mathbb{X}^* := L^{p/(p-1)}(\mathcal{O}).$$

The inner product in \mathbb{H} is given by

$$\langle x, y \rangle_{\mathbb{H}} := \int_{\mathcal{O}} (-\Delta)^{-1/2} x(\xi) \cdot (-\Delta)^{-1/2} y(\xi) d\xi, \quad x, y \in \mathbb{H} = W^{-1,2}(\mathcal{O}).$$

Note that $-\Delta$ establishes an isomorphism between $W_0^{1,2}(\mathcal{O})$ and $W^{-1,2}(\mathcal{O})$. We shall identify $W_0^{1,2}(\mathcal{O})$ with the dual space \mathbb{H}^* of \mathbb{H} , and hence $\mathbb{H}^* = W_0^{1,2}(\mathcal{O}) \subset L^{p/(p-1)}(\mathcal{O})$. Thus, we have the evolution triple

$$\mathbb{X} \subset \mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{X}^*$$

where \simeq is understood through $-\Delta$.

Let $\phi_p(r) := r|r|^{p-2/2}$, and define for $x \in \mathbb{X} = L^p(\mathcal{O})$

$$A(x) := \Delta \phi_p(x).$$

Then $A(x) \in \mathbb{X}^*$ and **(H1)**-**(H4)** hold(cf. [28, 21]).

Let $B_1, \dots, B_n \in L_2(\mathbb{U}_Q, \mathbb{H})$, and define

$$B(t, x) := \sum_{k=1}^n g_k([e_{n_1}, x]_{\mathbb{H}}, \dots, [e_{n_k}, x]_{\mathbb{H}}) B_k, \quad e_{n_j} \in \mathbb{H}, \quad (38)$$

where g_k are Lipschitz continuous functions on \mathbb{R}^{n_k} . Then such $B(t, x)$ satisfies **(H5)**. It should be noticed that if $\sigma \in C_b^\infty(\mathbb{R})$ is not linear, the mapping $x \mapsto \sigma(x)$ is in general not Lipschitz from $W^{-1,2}(\mathcal{O})$ to $W^{-1,2}(\mathcal{O})$.

Consider the following small perturbation of stochastic porous medium equation

$$\begin{cases} dX_\varepsilon(t) = \Delta(\phi_p(X_\varepsilon(t)))dt + \sqrt{\varepsilon}B(t, X_\varepsilon(t))dW(t), \\ X_\varepsilon(t, \xi) = 0, \quad \forall \xi \in \partial\mathcal{O}, \\ X_\varepsilon(0, \xi) = x(\xi) \in W^{-1,2}(\mathcal{O}). \end{cases} \quad (39)$$

Let ν_ε be the law of $X_\varepsilon(t)$ in $\mathbb{C}_T(\mathbb{H}) \cap L^p(0, T; \mathbb{X})$. Then the conclusion of Theorem 3.8 holds.

Acknowledgements:

The second named author would like to thank Professor Benjamin Goldys for providing him an excellent environment to work in the University of New South Wales. His work is supported by ARC Discovery grant DP0663153 of Australia. The authors also thanks Dr. Wei Liu for sending us his preprint [19].

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